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WEIBULL PREDICTION INTERVALS FOR FINITE LOTS WHEN TESTING OCCUR--ETC(U)
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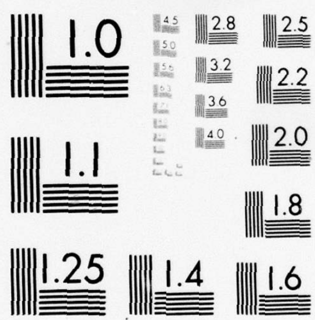

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Weibull Prediction Intervals for Finite Lots When Testing Occurs at Stress Levels Other than Nominal*

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Abstract

In this paper a two-parameter Weibull model is assumed for time to failure (or number of cycles to failure). The inverse power law is assumed to relate level of stress and the Weibull scale parameter with the Weibull shape parameter independent of stress level.

It is supposed that a life test results in Weibull failure-time data applying to the first r failures in a sample of size n. The procedure derived herein allows one to use these data to obtain an approximate lower confidence bound on the time of the ith failure in a lot of size m selected from the same population as the sample. The precision of the approximation is investigated for certain cases in which i = 1, that is, the first failure in a lot is of interest. A method is given for determining whether the approximation is sufficiently precise for use, and an example of use of the approximation is provided.

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1. Introduction

The following applies in certain cases in which life tests are performed on a sample at two or more levels of stress and inferences about a large, but finite, independent sample (lot) are to be made for a nominal stress level at which no tests are performed. A Weibull model is assumed for time to failure and the inverse power law is assumed to relate level of stress and the Weibull scale parameter δ.

Actually, the approach given here applies also when failure times have a lognormal distribution and the data are censored. It is applicable to both Weibull and lognormal data whenever some known polynomial function of degree k relates a specified function of level of stress and the logarithm of time to failure and when the scale parameter ξ (or σ) of the distribution of the logarithm of time to failure (inverse of the Weibull or lognormal shape parameter) is independent of stress level. See, for example, Charles [2] and Saunders [18].

The life tests from which the failure-time data are obtained may be conducted at any number ℓ of stress levels with the constraint, of course, $\ell \geq k+1$, allowing for estimation of the k+1 polynomial regression parameters. The total number of life tests conducted must be at least $\ell+1$ in order that at least two tests be made at one or more stress levels, thus insuring that ξ is estimable.

The procedure to be described allows one to obtain a small-sample approximate prediction interval, an approximate lower confidence bound, for the time to failure of the first (or second, third, etc.) failure in a finite lot of specified size. To obtain the prediction interval one must first calculate best (or approximately best) linear unbiased estimates, best (or approximately

best) linear invariant estimates, or maximum-likelihood estimates of the location and scale parameters of the distribution of the logarithm of time to failure. Details of the method are described in Sections 3 and 4, precision of the approximation is described in Section 5, and an example is siven in Section 6.

2. The Model

As noted above, we assume that failure time T has a Weibull distribution with shape parameter $\xi^{-1} = \alpha$ and scale parameter $\delta > 0$. Then $X = \ell nT$ has an extreme-value distribution (type-I distribution of the smallest extreme) given by

$$F_X(x) = 1 - \exp\{-\exp(x-\eta)/\xi\} \quad , \quad \xi > 0 \tag{1.1}$$

where $\eta = \ell n\delta$ is the distribution mode (a location parameter) and the scale parameter $\pi\xi/\sqrt{6}$ is the distribution standard deviation. The parameter ξ is fixed, but η varies with level of stress applied to induce failure.

We consider ordered observables $X_{1,n},\ldots,X_{r,n}$, $r \leq n$, $X_{i+1,n} \geq X_{i,n}$, $i = 1,\ldots,r$, from a sample randomly selected from a distribution given by (1.1). That is, we assume termination of the life test at time $t_{r,n} = \exp(x_{r,n})$. Let $X^*_{r,n}$ be defined as the linear combination of $X_{1,n},\ldots,X_{r,n}$ with smallest mean squared error in estimating its expectation, of the form $\eta + \kappa\xi$, $-\infty < \kappa < \infty$. It is thus the linear estimate most representative of the variate X (logarithm of failure time) associated with the corresponding η (stress level).

Mann [8] shows that $X^*_{r,n}$ is equal to $\eta^*_{r,n} - (B/C)\xi^*_{r,n}$ where $\eta^*_{r,n}$ and $\xi^*_{r,n}$ are best (minimum variance) among linear unbiased estimators of η and ξ, $C\xi^2$

is the variance of $\xi^*_{r,n}$ and $B\xi^2$ is the covariance of $\eta^*_{r,n}$ and $\xi^*_{r,n}$. The variance is $(A-B^2/C)\xi^2$, where $A\xi^2$ is the variance of $\eta^*_{r,n}$. Note that the covariance of $X^*_{r,n}$ and $\xi^*_{r,n}$ is $B\xi^2 - (B/C)C\xi^2 \equiv 0$. This is a property that will be important in constructing our prediction interval. Also note that for a complete sample from a normal distribution with mean μ and standard deviation σ, $X^*_{n,n}$ is equal to $\bar{X}_{n,n} \equiv \sum_{i=1}^{n} X_i/n$. This holds because the distribution is symmetric and hence the covariance of $\bar{X}_{n,n}$ and any unbiased linear estimator of σ is zero. We comment here that the results that follow apply in principle when X has a normal distribution by simply substituting μ for η and σ for ξ.

It is useful to mention at this point that tables of coefficients, for obtaining optimal or nearly optimal linear estimates of extreme-value or normal parameters along with values of A, B and C (or functions of these) are given by Mann [6,7], Mann, Schafer and Singpurwalla [14], Mann and Fertig [12], Sarhan and Greenberg [17], and Mann, Witherspoon and Fertig [15] as functions of r and n. Also, results in Lawless [4], Fertig and Mann [3], and Mann [10] indicate that maximum-likelihood estimates can be substituted for best linear invariant estimates in obtaining the approximations to be described herein. Best linear invariant estimators of η and ξ are $\tilde{\eta}_{r,n} = \eta_{r,n} - B\tilde{\xi}_{r,n}$ and $\tilde{\xi}_{r,n} = \tilde{\xi}^*_{r,n}/(1+C)$, respectively, so that $X^*_{r,n} \equiv \eta^*_{r,n} - (B/C)\xi^*_{r,n}$ is equal to $\tilde{\eta}_{r,n} - (B/C)\tilde{\xi}_{r,n}$. This means that the best unbiased and best invariant estimators of $E(X^*_{r,n}) = \eta - (B/C)\xi$ are equivalent. This is true of no other location parameter of the form $\eta + \kappa\xi$. Note that for the normal distribution best linear unbiased and best linear invariant estimators of $\mu + \kappa\sigma$ are not equivalent unless $\kappa = (-B/C) \equiv 0$.

The model we are assuming as relating the logarithm of failure time and a function of stress level s is polynomial. Since X_s^* is to represent the logarithm of failure time at stress level s, we have

$$E(X_s^*) = \beta_0 + \beta_1\{g(s)\} + \ldots + \beta_k\{g(s)\}^k \tag{2.1}$$

Using X_s^* allows for most efficient use of the data in estimating or predicting values of the parameter $\eta + \kappa\xi$ for any κ or of failure times at the nominal level. See, for example, tabulations given by Mann [11], in which comparisons are made of results based on (2.1) and those based on $E(X_s) = \eta_s - \gamma\xi_s = \sum_{j=0}^{k} \beta_j[g(s)]^k$ with γ equal to Euler's constant, approximately 0.577216.

The method to be described can be used even though only a single observation is made at one or more levels (as long as the total number of observations $\ell \geqslant k+1$). When $r = 1$ and the sample size at the αth stress level is n_α, then $X_{n\alpha}^*$ is equal to X_{1,n_α} with expectation $\eta_\alpha - \gamma\xi - \ell n n_\alpha \xi$, where the subxcript α indicates the appropriate value at the αth level. The variance of X_{1,n_α} is $\pi\xi^2/6$. Note that difficulties arise when any n_α is equal to one if one attempts to use a model directly relating g(s) and η, as one might attempt to do for the inverse power-law model. The difficulties occur because two observations are needed at each level to estimate η.

3. Calculation of Approximate Prediction Intervals from Nominal-Level Data

In [9], a method is described and investigated for obtaining from Weibull failure data a prediction interval for the first, second, etc., failure in a finite lot of specified size. We let $Y_{i,m}$ be the natural

logarithm of the time of the ith failure in a lot of size m. We assume a sample of size n has been randomly selected from the same population of logarithms of failure times from which the (sample) of size m has been or will be independently selected. Form $X^*_{r,n} = \eta^*_{r,n} - (B/C)\xi^*_{r,n}$, a linear function of $X_{1,n},\ldots,X_{r,n}$, all the available ordered observables of the log failure times, as described in Section 2. Then, $(X^* - Y_{i,m})/E(X^* - Y_{i,m})$ and ξ^* are shown in [9] to have zero covariance and each to be approximate chi-square-over-degrees-of-freedom variates. Since this is so, then under certain conditions, relating the two numbers of degrees of freedom,

$$F_y = (X^*_{r,n} - Y_{i,m})/[\xi^*_{r,n}\{-B/C - E(Z_{i,m})\}] \quad , \tag{3.2}$$

with $E(Z_{i,m}) = E\{(Y_{i,m} - \eta)/\xi\}$, might be expected to be approximately distributed as a classical F variate with degrees of freedom

$$\nu_1 = 2\{B/C + E(Z_{i,m})\}^2/\{A - B^2/C + \mathrm{Var}(Z_{i,m})\}$$

and

$$\nu_2 = 2/C \quad .$$

Values of $E(Z_{i,m})$ and the reduced variance $\mathrm{Var}(Z_{i,m})$ are published by White [19] for samples of size m, m = 2(1)50(5)100. Also, if i is small, it is relatively easy to calculate $E(Z_{i,m})$ from the expression given by Lieblein [5].

$$\sum_{\beta=1}^{i} (-1)^{\beta-1} n \binom{n-1}{i-1} \binom{i-1}{\beta-1} \frac{-\gamma-\ln(n-i+\beta)}{n-i+\beta} \quad . \tag{3.1}$$

A similar expression in [5] yields $\mathrm{Var}(Z_{i,m})$. For $i = 1$, $E(Z_{i,m}) = -\gamma-\ln nm$ and $\mathrm{Var}(Z_{i,m}) = \pi^2/6$. A method for calculating percentiles of an F distribution with noninteger degrees of freedom is given by Mann and Grubbs [13] and described in [14] and [9].

Combinations of values of ν_1 and ν_2 for which the F approximation to F_y is sufficiently precise for use with Weibull or lognormal data is given by Mann [9] and Fertig and Mann [3]. Note that ν_2 depends upon sample size n and censoring number r while ν_1 depends not only upon r and n, but also upon i and lot size m.

To obtain an approximate lower prediction limit for $Y_{i,m}$ at confidence level $1-\alpha$ from (3.2), one simply forms

$$\exp\left[X^*_{r,n} + F_{1-\alpha}(\nu_1,\nu_2)\{E(Z_{i,m}) + B/C\}\xi^*_{r,n}\right] \quad , \tag{3.3}$$

where $F_{1-\alpha}(\nu_1,\nu_2)$ is the $100(1-\alpha)$th percentile of the F distribution with ν_1 and ν_2 degrees of freedom. Because ν_1 tends to be small with respect to ν_2 here, precision of results based on the method described in [13] are greatly increased if one approximates $1/F_\alpha(\nu_2,\nu_1)$, which is equal to $F_{1-\alpha}(\nu_1,\nu_2)$, rather than approximating $F,\alpha\ (\nu_1,\nu_2)$ directly.

In Section 4, we describe how this result can be adapted to a situation in which life tests are conducted at levels other than nominal.

4. Nominal Level Prediction Intervals

Let us suppose first, as assumed by Mann [8], that the life tests are to be conducted at k+1 levels, where k specifies the degree of the polynomial relating $E(X_s^*)$ and g(s). Fixing the number of levels at k+1 facilitates the exposition of the analysis by allowing for expression of X_o^*, the best linear unbiased estimate of value of $E(X^*)$ at the nominal level s_o, in terms of the Legendre polynomial,

$$L_\alpha(h) = \frac{(h-h_1)\dots(h-h_{\alpha-1})(h-h_{\alpha+1})\dots(h-h_k)}{(h_\alpha-h_1)\dots(h_\alpha-h_{\alpha-1})(h_\alpha-h_{\alpha+1})\dots(h_\alpha-h_k)} \quad , \quad \alpha = 1,\dots,k+1$$

where $h = g(s)$ and $h_\alpha = g(s_\alpha)$. Subsequently we will treat the more general case in which the number of levels ℓ may be greater than k+1.

From Mann [8], one sees that under the assumption (2.1),

$$X_o^* = \sum_{\alpha=1}^{k+1} L_\alpha(h_o)X_\alpha^* \tag{4.1}$$

and that the best (minimum-variance) linear estimator ξ^* of ξ is given by

$$\xi^* = \sum_{\alpha=1}^{k+1} (\xi^*/C_\alpha) \Big/ \sum_{\alpha=1}^{k} (1/C_\alpha) \quad ,$$

with variance $C = \left[\sum_{\alpha=1}^{k+1} (1/C_\alpha)\right]^{-1}$. Here X_α^*, ξ^* and C_α are the values of X^*, ξ^* and C applying to the sample size, censoring number and the data at the αth level. The expectation of X_o^* is

$$E(X_o^*) = \left[\eta_o - \xi \sum_{\alpha=1}^{k+1} L_\alpha(h_o)(B_\alpha/C_\alpha)\right] \equiv [\eta_o - \xi(B/C)_o]$$

and the variance of X_o^* is given by

$$\mathrm{Var}(X_o^*) = \sum_{\alpha=1}^{k+1} \{L_\alpha(h_o)\}^2 (A_\alpha - B_\alpha^2/C_\alpha)\xi^2 \equiv \mathrm{Var}(Z_o^*)\xi^2 \quad ,$$

with

$$Z_o^* = (X_o^* - \eta_o)/\xi \quad .$$

It is important to recall that $\sum_{\alpha=1}^{k+1} L_\alpha(h_o) = 1$. This implies that $X_o^* - Y_{i,m} = \sum_{\alpha=1}^{k+1} L_\alpha(h_o)X^* - Y_{i,m}$ is equal to $\sum_{\alpha=1}^{k+1} L_\alpha(h_o)(X^* - Y_{i,m})$ and hence that $X_o^* - Y_{i,m}$ is a linear function of the approximate gamma variates $(X_\alpha^* - Y_{i,m})$, $\alpha = 1,\ldots,k+1$. Therefore, by results of Box [17] and Patnaik [16] $(X_o^* - Y_{i,m})/E(X_o^* - Y_{i,m})$ is approximately a chisquare over its degrees of freedom given by

$$\nu_1^o = 2\{(B/C)_o + E(Z_{i,m})\}^2/[\mathrm{Var}(Z_o^*) + \mathrm{Var}(Z_{i,m})]$$

The estimator ξ^* is also, clearly, a linear function of the ξ_α^*'s, each of which is approximately an independent chisquare over degrees of freedom and ξ^* is an unbiased estimator of ξ. Thus, ξ^*/ξ is approximately chisquare over degrees of freedom with degrees of freedom equal to

$$\nu_2^o = 2\sum_{\alpha=1}^{k+1} (1/C_\alpha) \quad .$$

As a result, one would expect that under appropriate conditions for ν_1^o and ν_2^o,

$$F_h = (X_o^* - Y_{i,m})/[-\xi^*\{(B/C)_o + E(Z_{i,m})\}] \tag{4.2}$$

is approximately distributed as an F-variate with ν_1^o and ν_2^o degrees of freedom. Consequently, a lower $(1-\alpha)$-level confidence bound on $T_{i,m} = \exp(Y_{i,m})$ is approximately given by

$$\exp[X_o^* + F_{1-\alpha}(\nu_1^o,\nu_2^o)\{E(Z_{i,m}) + (B/C)_o\}\xi^*] \quad . \tag{4.3}$$

If the number ℓ of testing levels is greater than k+1, then, the model cannot be expressed as in (4.1). Rather, we have,

$$E(X_o^*) = \underset{\sim}{H}\underset{\sim}{\beta} \tag{4.4}$$

where $\underset{\sim}{\beta}$ is the column vector $(\beta_o,\beta_1,\ldots,\beta_k)'$ and $\underset{\sim}{H}$ is a matrix of which the element in the αth row and jth column is equal to $h_\alpha^j = \{g(\sigma_\alpha)\}^j$, $\alpha = 1,\ldots,\ell$, $j = 0,1,\ldots,k$. Then, the best linear unbiased estimator vector for $\underset{\sim}{\beta}$ is

$$\underset{\sim}{\beta}^* = (\underset{\sim}{H}'\underset{\sim}{V}^{-1}\underset{\sim}{H})^{-1}\underset{\sim}{H}'\underset{\sim}{V}^{-1}\underset{\sim}{X}^* \quad , \tag{4.5}$$

where $\underset{\sim}{X}^*$ is the column vector $(X_1^*,\ldots,X_\ell^*)'$ and $\underset{\sim}{V}$ is the diagonal matrix with elements consisting of $(A_\alpha - B_\alpha^2/C_\alpha)$, $\alpha = 1,\ldots,\ell$. The estimator X_o^* is then given by the right side of (4.4) with $\underset{\sim}{\beta}^* = (\beta_o^*,\beta_1^*,\ldots,\beta_k^*)$ substituted for $\underset{\sim}{\beta}$. The expectation $-(B/C)_o$ is given by $\underset{\sim}{H}\underset{\sim}{W}$ where $\underset{\sim}{W}$ is the vector defined by the right side of (4.5), with $(B_1/C_1,\ldots,B_\ell/C_\ell)$ substituted for $(X_1^*,\ldots,X_\ell^*)$. The covariance matrix of the elements of $\underset{\sim}{\beta}^*$ is given by $(\underset{\sim}{H}'\underset{\sim}{V}^{-1}\underset{\sim}{H})^{-1}\xi^2$ so that the variance of Z_o^* is $\underset{\sim}{H}_o' \, (\underset{\sim}{H}'\underset{\sim}{V}^{-1}\underset{\sim}{H})^{-1}\underset{\sim}{H}_o$, where $H_o' = (1,g(s_o),\ldots,\{g(s_o)\}^k)$.

In the next section, we give results of a Monte Carlo study undertaken to determine the precision of the approximation (4.2).

5. Precision of the Approximation

A Monte Carlo study revealed that constraints on the precision of the approximation in [9] for Weibull failure data at a nominal level apply roughly if we modify the constraint on ν_2^o so that it depends upon the number ℓ of stress levels at which the life tests are conducted as well as upon sample sizes and censoring numbers. Thus, for confidence level equal to $1-\alpha$, the approximate F values are in error by less than one percent roughly for

$$1-\alpha = 0.90: \quad \nu_2^o \geqslant 8.88\ell, \ 0.15\nu_2^o + 4.7 < \nu_1^o < 1.9\nu_2^o - 1.2$$

$$1-\alpha = 0.95: \quad \nu_2^o \geqslant 8.88\ell, \ 0.27\nu_2 + 5.6 < \nu_1^o < 0.90 + 6.5$$

$$1-\alpha = 0.98: \quad \nu_2^o \geqslant 15.16\ell, \ 0.36\nu_2 + 2.3 < \nu_1 < 0.78\nu_2 + 4.1$$

$$\alpha = 1,\ldots,\ell$$

All the Monte Carlo results herein have been obtained using the approximation of Mann and Grubbs [13] with $F_{1-\alpha}(\nu_1,\nu_2)$ obtained as $[F_\alpha(\nu_2,\nu_1)]^{-1}$, as described in Section 4. When $n_\alpha = r_\alpha$, $\alpha = 1,\ldots,\ell$, $\nu_2^o \geqslant 8.88$ if $n_\alpha \geqslant 4$, $\alpha = 1,\ldots,\ell$ and $\nu_2^o \geqslant 15.16\ell$ if $n_\alpha \geqslant 6$, $\alpha = 1,\ldots,\ell$.

Recent unpublished results of Fertig and Mann suggest that a more generally applicable approximation to the distributions of (3.2) and (4.2)

are given by a logarithm of an F distribution divided by an independent chi-square variate. It is planned that percentiles of this distribution will be tabulated for various combinations of the three parameters.

In the 15 cases studied in the Monte Carlo analysis, $Y_{i,m}$ was taken as $Y_{1,m}$ and g(s) was taken to be $\ell n(s)$ (applicable to the inverse-power law model). A Monte Carlo sample size of 5000 was used and appeared adequate to insure, in all cases, precision of within about 1 percent at $1-\alpha = .98$.

It was found that although the approximation worked very well in general for k = 1, i.e., linear (inverse power-law) models when m was sufficiently large compared with $\sum_{\alpha=1}^{\ell} n_\alpha$, it did not perform well for k > 1 unless s_o was inside the range spanned by $s_1,\ldots,s_\ell$. For k > 1 and $s_o < s_1 < s_2 < \ldots < s_\ell$, the value of ν_1^o tends to be too small to satisfy the constraints given above. The precision of the approximation increases as the n_α's increase for values of $p_\alpha = r_\alpha/n_\alpha$ fixed and as the p_α's increase for fixed values of the n_α's. Increase in the value of the lot size m, for all else fixed, also increases the precision of the approximation. One would also expect some degradation in precision for increase in i with fixed m.

The value of ν_2^o is affected not only by the values of r_α and n_α, $\alpha = 1,\ldots,\ell$, and of i and m, but also by the relations of the stress levels among themselves and to the nominal level s_o and by the degree of the polynomial to be fitted.

Given in Table 1 are results of six of the Monte Carlo analyses. In each case, s_o is equal to 5 or 15, s_1 is equal to 15 or 5, $s_2 = 50$ and, if used, $s_3 = 100$. As noted earlier, the function g(s) was taken to be $\ell n(s)$.

Table 1. Approximate and Monte Carlo (M.C.) Values of *100*γth Percentiles of the Approximate F-Variate F_h: $g(s) = \ln(s)$

$\ell=2, k+1=2$; $\nu_0=5$; $\nu_1=15$; $\nu_2=50$

$n_1=r_1=6$; $\nu_1=11.0$; $i=1$ $n_2=r_2=7$; $\nu_2=33.5$; $m=18$			$n_1=r_1=6$: $\nu_1=13.0$: $i=1$ $n_2=r_2=7$; $\nu_2=33.5$; $m=25$		
γ	M.C. Percentile	Approximate Percentile	γ	M.C. Percentile	Approximate Percentile
0.75 0.90	1.32 1.73	1.33 1.77	0.75 0.90	1.32 1.73	1.32 1.73
0.95	2.04	2.09	0.95	2.00	2.02
0.98	2.47	2.41	0.98	2.37	2.42
0.99	2.81	2.84	0.99	2.69	2.71

$\ell=3$; $n_2=r_1=4$; $n_3=5$; $r_3=4$; $s_2=50$; $s_3=100$

$k+1=2$; $\nu_1=15.4$; $i=1$ $n_1=r_1=8$; $\nu_2=38.3$; $m=18$ $s_0=5$, $s_1=15$			$k+1=3$: $\nu_1=19.0$: $i=1$ $n_1=r_1=6$; $\nu_2=31.9$; $m=40$ $s_0=15$, $s_1=15$		
γ	M.C. Percentile	Approximate Percentile	γ	M.C. Percentile	Approximate Percentile
0.75	1.30	1.30	0.75	1.28	1.30
0.90	1.68	1.66	0.90	1.65	1.66
0.95	1.93	1.93	0.95	1.93	1.92
0.98	2.32	2.27	0.98	2.24	2.26
0.99	2.53	2.53	0.99	2.48	2.52

$\ell=2, k+1=2$; $s_0=5$; $s_1=15$; $s_2=50$

$n_1=r_1=4$; $\nu_1=10.4$; $i=1$ $n_2=5$; $r_2=4$; $\nu_2=16.8$; $m=18$			$n_1=r_1=6$; $\nu_1=7.74$; $i=1$ $n_2=r_2=7$; $\nu_2=33.5$; $m=10$		
γ	M.C. Percentile	Approximate Percentile	γ	M.C. Percentile	Approximate Percentile
0.75	1.42	1.43	0.75	1.36	1.36
0.90	2.03	2.01	0.90	1.85	1.86
0.95	2.46	2.48	0.95	2.17	2.24
0.98	3.11	3.16	0.98	2.55	2.74
0.99	3.68	3.73	0.99	2.93	3.31

6. Example

Two random samples of bonded composite joints were fatigue-to-failure tested at accelerated test levels. A Weibull distribution is known to be appropriate as a failure model. A sample of size 7 was subjected to test at stress level $s_1 = 20$ until four failures were observed, and $\eta^*_{4,7} = 5.51$ and $\xi^*_{4,7} = .064$ were calculated from the observed number of cycles to failure. A second sample of size 5 tested until all five specimens failed at $s_2 = 50$, yielded $\eta^*_{5,5} = 4.31$ and $\xi^*_{5,5} = .058$. The inverse power-law model was used to predict the time of the first failure in a lot of size 100 at the nominal level $s_0 = 10$.

From tabulations in [14], we find for $r_1 = 4$, $n_1 = 7$ that $A = 0.37265$, $B = 0.15696$ and $C = 0.28016$ so that $X^*_{4,7} = 5.47$. Similarly, from $A = 0.23140$, $B = -0.033991$ and $C = 0.16665$ for $r_2 = n_2 = 5$ we have $X^*_{5,5} = 4.32$.

Since $h_0 = \ell n10 = 2.302$, $h_1 = \ell n20 = 2.996$ and $h_2 = \ell n50 = 3.912$, then $L_1(h_0) = \frac{2.302 - 3.912}{2.996 - 3.912} = 1.758$ and $L_2(h_0) = \frac{2.302 - 2.996}{3.912 - 2.996} = -0.758$. Thus, $X^* = 1.758(5.47) - 0.758(4.32) = 6.34$. Also, $\xi^* = (\xi_{4,7}/0.28016 + \xi^*_{5,5}/0.16665)/(1/0.28016 + 1/0.16664) = .0602$ with variance $0.10449\xi^2$.

The expectation of $(X^*_0 - Y_{1,100})/\xi$ is then $-[1.758(0.56025) - 0.758(-0.20398) -0.577216 - 4.605] = 4.043$ and the variance of $(X^*_0 - Y_{1,100})/\xi$ is $1.758^2(0.37265 - 0.08794) + 0.758^2(0.23140 - 0.00693) + 1.6449 = 2.653^2$. Thus, the degrees of freedom for the approximate F-variate F_h are $\nu^o_1 = 2(4.043^2)/2.653 = 12.33$ and $\nu^o_2 = 2/0.10449 = 19.14$. Using the method of Mann and Grubbs [13] we approximate first the tenth percentile of a Beta distribution with parameters $19.14/2 = 9.57$ and $12.32/2 = 6.16$. This is accomplished by letting m equal $\ell n(9.57 + 6.16 = 0.5) - \ell n(9.57 - 0.5) = 0.5183$ and v equal $(9.57 - 0.5)^{-1} - (9.57 + 6.16 - 0.5)^{-1} = 0.4493$ in

$$\exp\left[-\,m\left(1-\frac{v}{9m^2}+1.282\sqrt{\frac{v}{9m^2}}\right)^3\right]=0.4993 \quad .$$

Then $F_{.10}(19.14,12.32) \cong (12.32/19.14)0.4493/(1 - 0.4493) = 0.525$ and $F_{.90}(12.32,19.14) = (0.525)^{-1} = 1.903$.

To calculate a prediction interval by use of (4.3) for the first failure in a lot of size 100, we need the values of

$$(B/C)_0 + E(Z_{1,100}) = -E(X_0^* - Y_{1,100})/\xi = -4.043 \quad ,$$

$\xi^* = 0.0602$ and $X_0^* = 6.34$. Then, from (4.3), a 90 percent prediction interval for the first failure in a lot of 100 at the nominal stress level, $s_0 = 10$, is given by

$$\exp[6.34 + 1.903(-4.043)0.0602] = \exp(5.88) = 357$$

numbers of cycles to failure.

Since ν_1 and ν_2 easily satisfy the conditions specified in Section 5 for precision of the approximation to be within 1 percent, the approximate prediction interval for $Y_{1,100}$ should be correct to within a unit in the second decimal place and the approximate prediction interval for $\exp(Y_{1,100})$ should be within 1 percent of the exact value.

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In this paper a two-parameter Weibull model is assumed for time to failure (or number of cycles to failure). The inverse power law is assumed to relate level of stress and the Weibull scale parameter with the Weibull shape parameter independent of stress level.

It is supposed that a life test results in Weibull failure-time data applying to the first r failures in a sample of size n. The procedure derived herein allows one to use these data to obtain an approximate lower confidence bound on the time of the ith failure in a lot of size m selected from the same (over)

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population as the sample. The precision of the approximation is investigated for certain cases in which i = 1, that is, the first failure in a lot is of interest. A method is given for determining whether the approximation is sufficiently precise for use, and an example of use of the approximation is provided.